Two point functions for the six vertex model with reflecting end

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June 25, 2010

Abstract

The two point functions, which give the probability that the spins turn down at the boundaries, are studied for the six vertex model on a $2N \times N$ lattice with domain wall boundary condition and left reflecting end. We consider two types of two point functions, and express them using determinants.

1 Introduction

The six vertex model is one of the most fundamental exactly solved models in statistical physics [1, 2, 3, 4]. Not only the periodic boundary condition but also the domain wall boundary condition is an interesting boundary condition. For example, the partition function is deeply related to the norm [5] and the scalar product [6] of the XXZ chain. The determinant formula of the partition function [7, 8] lead Slavnov [6] to obtain a compact representation of the scalar product, which plays a fundamental role in calculating correlation functions of the XXZ chain [9, 10, 11, 12]. The determinant formula also led to a deep advance in enumerative combinatorics [13, 14, 15]. For example, it was used to give a concise proof of the numbers of the alternating sign matrices for a given size. Recently, the correspondences between the partition function and the Schur polynomial [16] and KP τ function [17] have been revealed.

The calculation of correlation functions are also interesting in the domain wall boundary condition itself. Several kinds of them such as the boundary correlation functions [18, 19, 20, 21] and the emptiness formation probability [22] have been calculated. Some of them are shown to be expressed in determinant forms.

The mixed boundary conditions of the domain wall and reflecting boundary [23] has also been studied. The partition function [24] and several kinds of boundary one point functions [25, 26] are computed and expressed using determinants.

In this paper, we calculate boundary two point functions for the six vertex model on a $2N \times N$ lattice with the mixed boundary condition. The two point functions we consider give the probability that the spins turn down at the boundaries. We calculate two types of two point functions, and express them in terms of determinants.

The outline of this paper is as follows. In the next section, we define the six vertex model with mixed boundary condition. We consider two types of two point functions, and call them Type I and Type II, according to the boundaries the spins we consider are associated with. Type I is evaluated in section 3, Type II in section 4.

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2 Six vertex model

The six vertex model is a model in statistical mechanics, whose local states are associated with edges of a square lattice, which can take two values. The Boltzmann weights are assigned to its vertices, and each weight is determined by the configuration around a vertex. What plays the fundamental role is the R-matrix

$$R(\lambda)=\begin{pmatrix} a(\lambda) & 0 & 0 & 0 \\ 0 & b(\lambda) & c(\lambda) & 0 \\ 0 & c(\lambda) & b(\lambda) & 0 \\ 0 & 0 & 0 & a(\lambda) \end{pmatrix}, \tag{1}$$

where

$$a(\lambda)=1,\ b(\lambda)=\frac{\mathrm{sh}\lambda}{\mathrm{sh}(\lambda+\eta)},\ c(\lambda)=\frac{\mathrm{sh}\eta}{\mathrm{sh}(\lambda+\eta)}. \tag{2}$$

The R-matrix satisfies the Yang-Baxter equation

$$R_{12}(\lambda)R_{13}(\lambda+\mu)R_{23}(\mu)=R_{23}(\mu)R_{13}(\lambda+\mu)R_{12}(\lambda). \tag{3}$$

In this paper, we consider the six vertex model on a $2N\times N$ lattice depicted in Figure 1. At the upper and lower boundaries, the spins are aligned all up and all down respectively. At the right boundary, the boundary spins are up for odd rows and down for even rows. We set $L_{\alpha k}(\lambda_\alpha,\nu_k)=R_{\alpha k}(\lambda_\alpha-\nu_k-\eta/2)$. At the intersection of the α-th row and the k-th column, we associate the statistical weight $\sigma_\beta^2 L_{\beta k}(-\lambda_\beta,\nu_k)\sigma_\beta^2,\ \beta=(\alpha+1)/2$ for α odd and $L_{\beta k}^{t_\beta}(\lambda_\beta,\nu_k),\ \beta=\alpha/2$ for α even. Between the $(2\alpha-1)$-th and (2α)-th row, the boundary statistical weight

$$K_+(\lambda_\alpha)=\begin{pmatrix} \mathrm{sh}(\lambda_\alpha+\eta/2+\zeta_+) & 0 \\ 0 & \mathrm{sh}(-\lambda_\alpha-\eta/2+\zeta_+) \end{pmatrix}, \tag{4}$$

is associated at the left boundary. For later convenience, we denote $\{\lambda\}=\{\lambda_1,\lambda_2,\dots,\lambda_N\},\{\nu\}=$

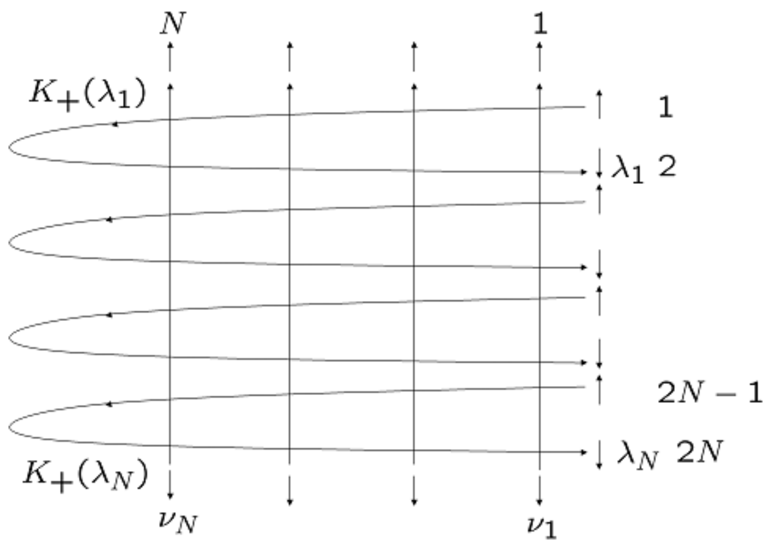

Figure 1: The six vertex model with left reflecting boundary.

$\{\nu_1, \nu_2, \dots, \nu_N\}$, and introduce the one-row monodromy matrix

$$
\begin{aligned}
T(\lambda_\alpha, \{\nu\}) =& L_{\alpha N}(\lambda_\alpha, \nu_N) \cdots L_{\alpha 1}(\lambda_\alpha, \nu_1) \\
=& \begin{pmatrix} A(\lambda_\alpha, \{\nu\}) & B(\lambda_\alpha, \{\nu\}) \\ C(\lambda_\alpha, \{\nu\}) & D(\lambda_\alpha, \{\nu\}) \end{pmatrix}.
\end{aligned} \tag{5}
$$

Combining two one-row monodromy matrices and the K-matrix (4), one can construct the double-row monodromy matrix

$$
\begin{aligned}
U^{t_\alpha}(\lambda_\alpha, \{\nu\}) &= T^{t_\alpha}(\lambda_\alpha, \{\nu\}) K_+(\lambda_\alpha) \sigma_\alpha^2 T(-\lambda_\alpha, \{\nu\}) \sigma_\alpha^2, \\
&= \begin{pmatrix} \mathcal{A}(\lambda_\alpha, \{\nu\}) & \mathcal{C}(\lambda_\alpha, \{\nu\}) \\ \mathcal{B}(\lambda_\alpha, \{\nu\}) & \mathcal{D}(\lambda_\alpha, \{\nu\}) \end{pmatrix}.
\end{aligned} \tag{6}
$$

The partition function of the six vertex model with mixed boundary condition, which is the summation of products of statistical weights over all possible configurations, can be represented as

$$
Z_{2N\times N}(\{\lambda\}, \{\nu\}) = w_N^- \mathcal{B}(\lambda_N, \{\nu\}) \cdots \mathcal{B}(\lambda_1, \{\nu\}) w_N^+, \tag{7}
$$

where $w_N^+ = \prod_{k=1}^N \uparrow_k$ and $w_N^- = \prod_{k=1}^N \downarrow_k$. It has the following determinant form [24]

$$
Z_{2N\times N}(\{\lambda\}, \{\nu\}) = \frac{\prod_{j=1}^N \prod_{k=1}^N \left[\mathrm{sh}^2(\nu_j + \eta/2) - \mathrm{sh}^2\lambda_k\right] \det_N \chi(\{\lambda\}, \{\nu\})}{\prod_{1\le j<k\le N} \left[\mathrm{sh}^2\nu_j - \mathrm{sh}^2\nu_k\right] \prod_{1\le j<k\le N} \left[\mathrm{sh}^2\lambda_k - \mathrm{sh}^2\lambda_j\right]}, \tag{8}
$$

where χ is an $N \times N$ matrix whose elements are given by

$$
\chi_{jk} = \chi(\lambda_j, \nu_k), \tag{9}
$$

$$
\chi(\lambda, \nu) = \frac{-\mathrm{sh}\eta\,\mathrm{sh}(2\lambda + \eta)\mathrm{sh}(\nu + \zeta_+)}{\left[\mathrm{sh}^2(\nu + \eta/2) - \mathrm{sh}^2\lambda\right]\left[\mathrm{sh}^2(\nu - \eta/2) - \mathrm{sh}^2\lambda\right]}. \tag{10}
$$

3 Type I

In this section, we consider the following two point function which we call Type I.

$$
\Psi_1(M, L) = \frac{\psi_1(M, L)}{Z_{2N\times N}(\{\lambda\}, \{\nu\})}, \tag{11}
$$

where

$$
\begin{aligned}
\psi_1(M, L) =& w_N^- \mathcal{B}(\lambda_N, \{\nu\}) \cdots \mathcal{B}(\lambda_{L+1}, \{\nu\}) \mathcal{G}_2(\lambda_L, \{\nu\}) \mathcal{B}(\lambda_{L-1}, \{\nu\}) \cdots \mathcal{B}(\lambda_{M+1}, \{\nu\}) \\
&\times \mathcal{F}(\lambda_M, \{\nu\}) \mathcal{B}(\lambda_{M-1}, \{\nu\}) \cdots \mathcal{B}(\lambda_1, \{\nu\}) w_N^-,
\end{aligned} \tag{12}
$$

$$
\mathcal{F}(\lambda_\alpha, \{\nu\}) =\downarrow_{2\alpha} W(\lambda_\alpha, \{\nu\}) \uparrow_{2\alpha-1}, \tag{13}
$$

$$
W(\lambda_\alpha, \{\nu\}) = T^{t_\alpha}(\lambda_\alpha, \{\nu\}) K_+(\lambda_\alpha) \frac{1}{2}(1 - \sigma_1^3) \sigma_\alpha^2 T(-\lambda_\alpha, \{\nu\}) \sigma_\alpha^2 \frac{1}{2}(1 + \sigma_1^3), \tag{14}
$$

$$
\mathcal{G}_2(\lambda_\alpha, \{\nu\}) =\downarrow_{2\alpha} X_2(\lambda_\alpha, \{\nu\}) \uparrow_{2\alpha-1}, \tag{15}
$$

$$
X_2(\lambda_\alpha, \{\nu\}) = \frac{1}{2}(1 - \sigma_2^3) T^{t_\alpha}(\lambda_\alpha, \{\nu\}) \frac{1}{2}(1 + \sigma_2^3) K_+(\lambda_\alpha) \sigma_\alpha^2 T(-\lambda_\alpha, \{\nu\}) \sigma_\alpha^2. \tag{16}
$$

This two point function is depicted in Figure 2, and gives the probability that the spin on the first column is turned down just on the $(2M-1)$-th row, and the spin on the second column is turned down just on the $(2L)$-th row. We calculate this two point function in several steps, and show that

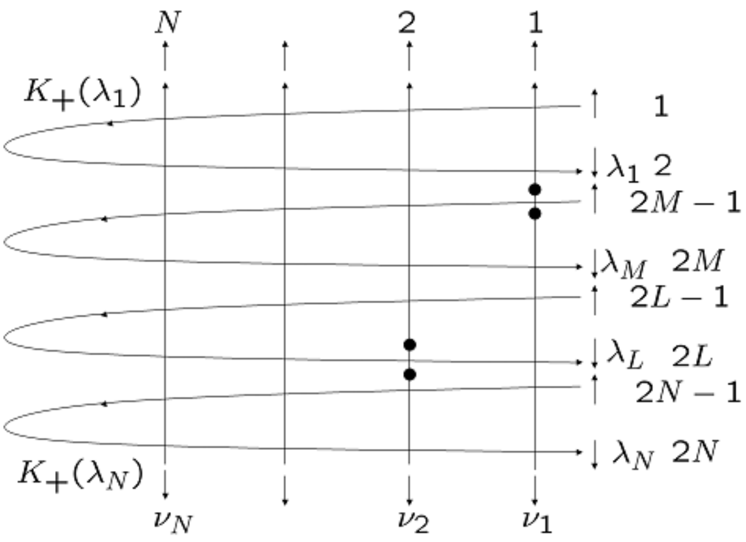

Figure 2: Type I (11).

as in [21], it can be expressed utilizing a determinant whose entries include differential operators. First, from the graphical representation, we find the numerator $\psi_1(M,L)$ becomes

$$\begin{aligned}\psi_1(M,L) =& -c(-\lambda_M-\nu_1-\eta/2)\prod_{j=1}^{M-1}\{b(-\lambda_j-\nu_1-\eta/2)b(\lambda_j-\nu_1-\eta/2)\}\\ &\times w_{N-1}^{-}\mathcal{B}(\lambda_N,\{\nu\}\backslash\nu_1)\cdots\mathcal{B}(\lambda_{L+1},\{\nu\}\backslash\nu_1)\mathcal{G}(\lambda_L,\{\nu\}\backslash\nu_1)\\ &\times\mathcal{B}(\lambda_{L-1},\{\nu\}\backslash\nu_1)\cdots\mathcal{B}(\lambda_{M+1},\{\nu\}\backslash\nu_1)\mathcal{D}(\lambda_M,\{\nu\}\backslash\nu_1)\\ &\times\mathcal{B}(\lambda_{M-1},\{\nu\}\backslash\nu_1)\cdots\mathcal{B}(\lambda_1,\{\nu\}\backslash\nu_1)w_{N-1}^{+},\end{aligned}\tag{17}$$

where

$$\mathcal{G}(\lambda_\alpha,\{\nu\}\backslash\nu_1)=\downarrow_{2\alpha}X(\lambda_\alpha,\{\nu\}\backslash\nu_1)\uparrow_{2\alpha-1},\tag{18}$$

$$X(\lambda_\alpha,\{\nu\}\backslash\nu_1)=\frac{1}{2}(1-\sigma_2^3)T^{t_\alpha}(\lambda_\alpha,\{\nu\}\backslash\nu_1)\frac{1}{2}(1+\sigma_2^3)K_+(\lambda_\alpha)\sigma_\alpha^2T(-\lambda_\alpha,\{\nu\}\backslash\nu_1)\sigma_\alpha^2.\tag{19}$$

Next, we need the action of the operator $\mathcal{D}$ on the vectors created by $\mathcal{B}$ operators. Utilizing the following formula

$$\begin{aligned}&\mathcal{D}(\lambda_i,\{\nu\})\mathcal{B}(\lambda_{i-1},\{\nu\})\cdots\mathcal{B}(\lambda_1,\{\nu\})w_N^+\\ &=\sum_{k=1}^{i}\frac{\mathrm{sh}\eta\,\mathrm{sh}(2\lambda_k+\eta)}{\mathrm{sh}2\lambda_k}\Bigg\{\frac{\mathrm{sh}(\lambda_k+\lambda_i)\mathrm{sh}(\lambda_k+\eta/2-\zeta_+)}{\mathrm{sh}^2\lambda_i-\mathrm{sh}^2(\lambda_k+\eta)}\prod_{j=1}^{N}\frac{\mathrm{sh}(\lambda_k-\nu_j-\eta/2)}{\mathrm{sh}(\lambda_k-\nu_j+\eta/2)}\prod_{\substack{j=1\\ j\neq k}}^{i}\frac{\mathrm{sh}^2(\lambda_k+\eta)-\mathrm{sh}^2\lambda_j}{\mathrm{sh}^2\lambda_k-\mathrm{sh}^2\lambda_j}\\ &\quad+\frac{\mathrm{sh}(\lambda_k-\lambda_i)\mathrm{sh}(-\lambda_k+\eta/2-\zeta_+)}{\mathrm{sh}^2\lambda_i-\mathrm{sh}^2(\lambda_k-\eta)}\prod_{j=1}^{N}\frac{\mathrm{sh}(-\lambda_k-\nu_j-\eta/2)}{\mathrm{sh}(-\lambda_k-\nu_j+\eta/2)}\prod_{\substack{j=1\\ j\neq k}}^{i}\frac{\mathrm{sh}^2(\lambda_k-\eta)-\mathrm{sh}^2\lambda_j}{\mathrm{sh}^2\lambda_k-\mathrm{sh}^2\lambda_j}\Bigg\}\\ &\quad\times\mathcal{B}(\lambda_i,\{\nu\})\cdots\check{\mathcal{B}}(\lambda_k,\{\nu\})\cdots\mathcal{B}(\lambda_1,\{\nu\})w_N^+,\end{aligned}\tag{20}$$

one has

$$
\begin{aligned}
\psi_1(M,L) =& -c(-\lambda_M-\nu_1-\eta/2)\prod_{j=1}^{M-1}\{b(-\lambda_j-\nu_1-\eta/2)b(\lambda_j-\nu_1-\eta/2)\}\sum_{\beta=1}^{M}\frac{\mathrm{sh}\eta\,\mathrm{sh}(2\lambda_\beta+\eta)}{\mathrm{sh}2\lambda_\beta}\\
&\times\Big\{\frac{\mathrm{sh}(\lambda_\beta+\lambda_M)\mathrm{sh}(\lambda_\beta+\eta/2-\zeta_+)}{\mathrm{sh}^2\lambda_M-\mathrm{sh}^2(\lambda_\beta+\eta)}\prod_{j=2}^{N}\frac{\mathrm{sh}(\lambda_\beta-\nu_j-\eta/2)}{\mathrm{sh}(\lambda_\beta-\nu_j+\eta/2)}\prod_{\substack{j=1\\ j\neq\beta}}^{M}\frac{\mathrm{sh}^2(\lambda_\beta+\eta)-\mathrm{sh}^2\lambda_j}{\mathrm{sh}^2\lambda_\beta-\mathrm{sh}^2\lambda_j}\\
&+\frac{\mathrm{sh}(\lambda_\beta-\lambda_M)\mathrm{sh}(-\lambda_\beta+\eta/2-\zeta_+)}{\mathrm{sh}^2\lambda_M-\mathrm{sh}^2(\lambda_\beta-\eta)}\prod_{j=2}^{N}\frac{\mathrm{sh}(-\lambda_\beta-\nu_j-\eta/2)}{\mathrm{sh}(-\lambda_\beta-\nu_j+\eta/2)}\prod_{\substack{j=1\\ j\neq\beta}}^{M}\frac{\mathrm{sh}^2(\lambda_\beta-\eta)-\mathrm{sh}^2\lambda_j}{\mathrm{sh}^2\lambda_\beta-\mathrm{sh}^2\lambda_j}\Big\}\\
&\times w_{N-1}^{-}\mathcal{B}(\lambda_N,\{\nu\}\backslash\nu_1)\cdots\mathcal{B}(\lambda_{L+1},\{\nu\}\backslash\nu_1)\mathcal{G}(\lambda_L,\{\nu\}\backslash\nu_1)\\
&\times\mathcal{B}(\lambda_{L-1},\{\nu\}\backslash\nu_1)\cdots\check{\mathcal{B}}(\lambda_\beta,\{\nu\}\backslash\nu_1)\cdots\mathcal{B}(\lambda_1,\{\nu\}\backslash\nu_1)w_{N-1}^{+},
\end{aligned}
\tag{21}
$$

getting rid of the operator $\mathcal{D}$. We repeat the similar procedure we did as above to (21). From the graphical description, one obtains

$$
\begin{aligned}
\psi_1(M,L) =& -c(-\lambda_M-\nu_1-\eta/2)\prod_{j=1}^{M-1}\{b(-\lambda_j-\nu_1-\eta/2)b(\lambda_j-\nu_1-\eta/2)\}b(-\lambda_L-\nu_2-\eta/2)\\
&\times c(\lambda_L-\nu_2-\eta/2)\sum_{\beta=1}^{M}\prod_{\substack{j=1\\ j\neq\beta}}^{L-1}\{b(-\lambda_j-\nu_2-\eta/2)b(\lambda_j-\nu_2-\eta/2)\}\frac{\mathrm{sh}\eta\,\mathrm{sh}(2\lambda_\beta+\eta)}{\mathrm{sh}2\lambda_\beta}\\
&\times\Big\{\frac{\mathrm{sh}(\lambda_\beta+\lambda_M)\mathrm{sh}(\lambda_\beta+\eta/2-\zeta_+)}{\mathrm{sh}^2\lambda_M-\mathrm{sh}^2(\lambda_\beta+\eta)}\prod_{j=2}^{N}\frac{\mathrm{sh}(\lambda_\beta-\nu_j-\eta/2)}{\mathrm{sh}(\lambda_\beta-\nu_j+\eta/2)}\prod_{\substack{j=1\\ j\neq\beta}}^{M}\frac{\mathrm{sh}^2(\lambda_\beta+\eta)-\mathrm{sh}^2\lambda_j}{\mathrm{sh}^2\lambda_\beta-\mathrm{sh}^2\lambda_j}\\
&+\frac{\mathrm{sh}(\lambda_\beta-\lambda_M)\mathrm{sh}(-\lambda_\beta+\eta/2-\zeta_+)}{\mathrm{sh}^2\lambda_M-\mathrm{sh}^2(\lambda_\beta-\eta)}\prod_{j=2}^{N}\frac{\mathrm{sh}(-\lambda_\beta-\nu_j-\eta/2)}{\mathrm{sh}(-\lambda_\beta-\nu_j+\eta/2)}\prod_{\substack{j=1\\ j\neq\beta}}^{M}\frac{\mathrm{sh}^2(\lambda_\beta-\eta)-\mathrm{sh}^2\lambda_j}{\mathrm{sh}^2\lambda_\beta-\mathrm{sh}^2\lambda_j}\Big\}\\
&\times w_{N-2}^{-}\mathcal{B}(\lambda_N,\{\nu\}\backslash\{\nu_1,\nu_2\})\cdots\mathcal{B}(\lambda_{L+1},\{\nu\}\backslash\{\nu_1,\nu_2\})\mathcal{A}(\lambda_L,\{\nu\}\backslash\{\nu_1,\nu_2\})\\
&\times\mathcal{B}(\lambda_{L-1},\{\nu\}\backslash\{\nu_1,\nu_2\})\cdots\check{\mathcal{B}}(\lambda_\beta,\{\nu\}\backslash\{\nu_1,\nu_2\})\cdots\mathcal{B}(\lambda_1,\{\nu\}\backslash\{\nu_1,\nu_2\})w_{N-2}^{+}.
\end{aligned}
\tag{22}
$$

Then, applying the formula

$$
\begin{aligned}
&\mathcal{A}(\lambda_i,\{\nu\})\mathcal{B}(\lambda_{i-1},\{\nu\})\cdots\mathcal{B}(\lambda_1,\{\nu\})w_N^{+}\\
=&\sum_{k=1}^{i}\frac{\mathrm{sh}\eta\,\mathrm{sh}(2\lambda_k+\eta)}{\mathrm{sh}2\lambda_k}\Big\{\frac{\mathrm{sh}(\lambda_k+\eta/2-\zeta_+)}{\mathrm{sh}(\lambda_k+\lambda_i+\eta)}\prod_{j=1}^{N}\frac{\mathrm{sh}(\lambda_k-\nu_j-\eta/2)}{\mathrm{sh}(\lambda_k-\nu_j+\eta/2)}\prod_{\substack{j=1\\ j\neq k}}^{i}\frac{\mathrm{sh}^2(\lambda_k+\eta)-\mathrm{sh}^2\lambda_j}{\mathrm{sh}^2\lambda_k-\mathrm{sh}^2\lambda_j}\\
&+\frac{\mathrm{sh}(-\lambda_k+\eta/2-\zeta_+)}{\mathrm{sh}(\lambda_k-\lambda_i-\eta)}\prod_{j=1}^{N}\frac{\mathrm{sh}(-\lambda_k-\nu_j-\eta/2)}{\mathrm{sh}(-\lambda_k-\nu_j+\eta/2)}\prod_{\substack{j=1\\ j\neq k}}^{i}\frac{\mathrm{sh}^2(\lambda_k-\eta)-\mathrm{sh}^2\lambda_j}{\mathrm{sh}^2\lambda_k-\mathrm{sh}^2\lambda_j}\Big\}\\
&\times\mathcal{B}(\lambda_i,\{\nu\})\cdots\check{\mathcal{B}}(\lambda_k,\{\nu\})\cdots\mathcal{B}(\lambda_1,\{\nu\})w_N^{+},
\end{aligned}
\tag{23}
$$

we have

$$\psi_1(M,L) = -\,c(-\lambda_M-\nu_1-\eta/2)\prod_{j=1}^{M-1}\{b(-\lambda_j-\nu_1-\eta/2)b(\lambda_j-\nu_1-\eta/2)\}b(-\lambda_L-\nu_2-\eta/2)$$

$$\times\, c(\lambda_L-\nu_2-\eta/2)\sum_{\beta=1}^{M}\prod_{\substack{j=1\\ j\neq\beta}}^{L-1}\{b(-\lambda_j-\nu_2-\eta/2)b(\lambda_j-\nu_2-\eta/2)\}\frac{\mathrm{sh}\eta\,\mathrm{sh}(2\lambda_\beta+\eta)}{\mathrm{sh}2\lambda_\beta}$$

$$\times\Big\{\frac{\mathrm{sh}(\lambda_\beta+\lambda_M)\mathrm{sh}(\lambda_\beta+\eta/2-\zeta_+)}{\mathrm{sh}^2\lambda_M-\mathrm{sh}^2(\lambda_\beta+\eta)}\prod_{j=2}^{N}\frac{\mathrm{sh}(\lambda_\beta-\nu_j-\eta/2)}{\mathrm{sh}(\lambda_\beta-\nu_j+\eta/2)}\prod_{\substack{j=1\\ j\neq\beta}}^{M}\frac{\mathrm{sh}^2(\lambda_\beta+\eta)-\mathrm{sh}^2\lambda_j}{\mathrm{sh}^2\lambda_\beta-\mathrm{sh}^2\lambda_j}$$

$$+\frac{\mathrm{sh}(\lambda_\beta-\lambda_M)\mathrm{sh}(-\lambda_\beta+\eta/2-\zeta_+)}{\mathrm{sh}^2\lambda_M-\mathrm{sh}^2(\lambda_\beta-\eta)}\prod_{j=2}^{N}\frac{\mathrm{sh}(-\lambda_\beta-\nu_j-\eta/2)}{\mathrm{sh}(-\lambda_\beta-\nu_j+\eta/2)}\prod_{\substack{j=1\\ j\neq\beta}}^{M}\frac{\mathrm{sh}^2(\lambda_\beta-\eta)-\mathrm{sh}^2\lambda_j}{\mathrm{sh}^2\lambda_\beta-\mathrm{sh}^2\lambda_j}\Big\}$$

$$\times\sum_{\substack{\alpha=1\\ \alpha\neq\beta}}^{L}\frac{\mathrm{sh}\eta\,\mathrm{sh}(2\lambda_\alpha+\eta)}{\mathrm{sh}2\lambda_\alpha}\Big\{\frac{\mathrm{sh}(\lambda_\alpha+\eta/2-\zeta_+)}{\mathrm{sh}(\lambda_\alpha+\lambda_L+\eta)}\prod_{j=3}^{N}\frac{\mathrm{sh}(\lambda_\alpha-\nu_j-\eta/2)}{\mathrm{sh}(\lambda_\alpha-\nu_j+\eta/2)}\prod_{\substack{j=1\\ j\neq\alpha,\beta}}^{L}\frac{\mathrm{sh}^2(\lambda_\alpha+\eta)-\mathrm{sh}^2\lambda_j}{\mathrm{sh}^2\lambda_\alpha-\mathrm{sh}^2\lambda_j}$$

$$+\frac{\mathrm{sh}(-\lambda_\alpha+\eta/2-\zeta_+)}{\mathrm{sh}(\lambda_\alpha-\lambda_L-\eta)}\prod_{j=3}^{N}\frac{\mathrm{sh}(-\lambda_\alpha-\nu_j-\eta/2)}{\mathrm{sh}(-\lambda_\alpha-\nu_j+\eta/2)}\prod_{\substack{j=1\\ j\neq\alpha,\beta}}^{L}\frac{\mathrm{sh}^2(\lambda_\alpha-\eta)-\mathrm{sh}^2\lambda_j}{\mathrm{sh}^2\lambda_\alpha-\mathrm{sh}^2\lambda_j}\Big\}$$

$$\times\, Z_{2(N-2)\times(N-2)}(\{\lambda\}\backslash\{\lambda_\alpha,\lambda_\beta\},\{\nu\}\backslash\{\nu_1,\nu_2\}). \tag{24}$$

Dividing (24) by the partition function (8) and simplifying, one gets

$$\Psi_1(M,L)$$

$$=\frac{\mathrm{sh}^2\eta}{\det_N\chi(\{\lambda\},\{\nu\})\mathrm{sh}(\lambda_M+\nu_1-\eta/2)\mathrm{sh}(\lambda_L-\nu_2-\eta/2)\prod_{j=1}^{M-1}\left[\mathrm{sh}^2(\nu_1-\eta/2)-\mathrm{sh}^2\lambda_j\right]}$$

$$\times\frac{\prod_{j=2}^{N}\left[\mathrm{sh}^2\nu_1-\mathrm{sh}^2\nu_j\right]\prod_{j=3}^{N}\left[\mathrm{sh}^2\nu_2-\mathrm{sh}^2\nu_j\right]}{\prod_{j=M}^{N}\left[\mathrm{sh}^2(\nu_1+\eta/2)-\mathrm{sh}^2\lambda_j\right]\prod_{j=1}^{L}\left[\mathrm{sh}^2(\nu_2-\eta/2)-\mathrm{sh}^2\lambda_j\right]\prod_{j=L+1}^{N}\left[\mathrm{sh}^2(\nu_2+\eta/2)-\mathrm{sh}^2\lambda_j\right]}$$

$$\times\sum_{\alpha=1}^{L}\sum_{\beta=1}^{M}(-1)^{\alpha+\beta}\epsilon_{\alpha\beta}\det_{N-2}\chi(\{\lambda\}\backslash\{\lambda_\alpha,\lambda_\beta\},\{\nu\}\backslash\{\nu_1,\nu_2\})\sum_{i,j=1}^{2}F_{i,j}(\lambda_\alpha,\lambda_\beta), \tag{25}$$

where

$$\epsilon_{\alpha\beta}=\begin{cases}1 & \alpha>\beta\\ 0 & \alpha=\beta\\ -1 & \alpha<\beta\end{cases}, \tag{26}$$

and

$$F_{1,1}(\lambda_\alpha,\lambda_\beta)=\frac{G_1(\lambda_\alpha)H_1(\lambda_\beta)}{\mathrm{sh}^2(\lambda_\alpha+\eta)-\mathrm{sh}^2\lambda_\beta}, \tag{27}$$

$$F_{1,2}(\lambda_\alpha,\lambda_\beta)=\frac{G_1(\lambda_\alpha)H_2(\lambda_\beta)}{\mathrm{sh}^2(\lambda_\alpha+\eta)-\mathrm{sh}^2\lambda_\beta}, \tag{28}$$

$$F_{2,1}(\lambda_\alpha,\lambda_\beta)=\frac{G_2(\lambda_\alpha)H_1(\lambda_\beta)}{\mathrm{sh}^2(\lambda_\alpha-\eta)-\mathrm{sh}^2\lambda_\beta}, \tag{29}$$

$$F_{2,2}(\lambda_\alpha,\lambda_\beta)=\frac{G_2(\lambda_\alpha)H_2(\lambda_\beta)}{\mathrm{sh}^2(\lambda_\alpha-\eta)-\mathrm{sh}^2\lambda_\beta}, \tag{30}$$

where $G_1(\lambda_\alpha), G_2(\lambda_\alpha), H_1(\lambda_\beta)$ and $H_2(\lambda_\beta)$ are

$$G_1(\lambda_\alpha) = \frac{\mathrm{sh}(\lambda_\alpha + \eta/2 - \zeta_+)}{\mathrm{sh}2\lambda_\alpha \mathrm{sh}(\lambda_\alpha + \lambda_L + \eta)} \frac{\prod_{j=1}^{L}\left[\mathrm{sh}^2(\lambda_\alpha+\eta) - \mathrm{sh}^2\lambda_j\right]\prod_{j=L+1}^{N}\left[\mathrm{sh}^2\lambda_\alpha - \mathrm{sh}^2\lambda_j\right]}{\prod_{j=3}^{N}\left[\mathrm{sh}^2\nu_j - \mathrm{sh}^2(\lambda_\alpha+\eta/2)\right]}, \tag{31}$$

$$G_2(\lambda_\alpha) = \frac{\mathrm{sh}(2\lambda_\alpha+\eta)\mathrm{sh}(\lambda_\alpha - \eta/2 + \zeta_+)}{\mathrm{sh}2\lambda_\alpha \mathrm{sh}(2\lambda_\alpha - \eta)\mathrm{sh}(\lambda_\alpha - \lambda_L - \eta)} \frac{\prod_{j=1}^{L}\left[\mathrm{sh}^2(\lambda_\alpha-\eta) - \mathrm{sh}^2\lambda_j\right]\prod_{j=L+1}^{N}\left[\mathrm{sh}^2\lambda_\alpha - \mathrm{sh}^2\lambda_j\right]}{\prod_{j=3}^{N}\left[\mathrm{sh}^2\nu_j - \mathrm{sh}^2(\lambda_\alpha-\eta/2)\right]}, \tag{32}$$

$$\begin{aligned} H_1(\lambda_\beta) =& \frac{\mathrm{sh}(\lambda_\beta+\lambda_M)\mathrm{sh}(\lambda_\beta+\eta/2-\zeta_+)\mathrm{sh}(\eta/2-\lambda_\beta-\nu_2)}{\mathrm{sh}2\lambda_\beta\mathrm{sh}(\lambda_\beta+\nu_2+\eta/2)} \\ &\times \frac{\prod_{j=1}^{M-1}\left[\mathrm{sh}^2(\lambda_\beta+\eta)-\mathrm{sh}^2\lambda_j\right]\prod_{j=M+1}^{N}\left[\mathrm{sh}^2\lambda_\beta-\mathrm{sh}^2\lambda_j\right]}{\prod_{j=3}^{N}\left[\mathrm{sh}^2\nu_j-\mathrm{sh}^2(\lambda_\beta+\eta/2)\right]}, \end{aligned} \tag{33}$$

$$\begin{aligned} H_2(\lambda_\beta) =& \frac{\mathrm{sh}(2\lambda_\beta+\eta)\mathrm{sh}(\lambda_\beta-\lambda_M)\mathrm{sh}(-\lambda_\beta+\eta/2-\zeta_+)\mathrm{sh}(\lambda_\beta-\nu_2+\eta/2)}{\mathrm{sh}2\lambda_\beta\mathrm{sh}(2\lambda_\beta-\eta)\mathrm{sh}(\lambda_\beta-\nu_2-\eta/2)} \\ &\times \frac{\prod_{j=1}^{M-1}\left[\mathrm{sh}^2(\lambda_\beta-\eta)-\mathrm{sh}^2\lambda_j\right]\prod_{j=M+1}^{N}\left[\mathrm{sh}^2\lambda_\beta-\mathrm{sh}^2\lambda_j\right]}{\prod_{j=3}^{N}\left[\mathrm{sh}^2\nu_j-\mathrm{sh}^2(\lambda_\beta-\eta/2)\right]}. \end{aligned} \tag{34}$$

Let us set λ_j as $\lambda_j = \lambda + z_j$. Note that the sum in (25) can be formally extended to N since $G_1(\lambda_\alpha) = G_2(\lambda_\alpha) = 0$ for $\alpha = L+1, \dots, N$ and $H_1(\lambda_\beta) = H_2(\lambda_\beta) = 0$ for $\beta = M+1, \dots, N$. Then, one finds that (25) can be expressed in the determinant form as

$$\begin{aligned} &\Psi_1(M,L) \\ =& \frac{\mathrm{sh}^2\eta}{\det_N \chi(\{\lambda\},\{\nu\})\mathrm{sh}(\lambda_M+\nu_1-\eta/2)\mathrm{sh}(\lambda_L-\nu_2-\eta/2)\prod_{j=1}^{M-1}\left[\mathrm{sh}^2(\nu_1-\eta/2)-\mathrm{sh}^2\lambda_j\right]} \\ &\times \frac{\prod_{j=2}^{N}\left[\mathrm{sh}^2\nu_1-\mathrm{sh}^2\nu_j\right]\prod_{j=3}^{N}\left[\mathrm{sh}^2\nu_2-\mathrm{sh}^2\nu_j\right]}{\prod_{j=M}^{N}\left[\mathrm{sh}^2(\nu_1+\eta/2)-\mathrm{sh}^2\lambda_j\right]\prod_{j=1}^{L}\left[\mathrm{sh}^2(\nu_2-\eta/2)-\mathrm{sh}^2\lambda_j\right]\prod_{j=L+1}^{N}\left[\mathrm{sh}^2(\nu_2+\eta/2)-\mathrm{sh}^2\lambda_j\right]} \\ &\times \det(\exp(z_j\partial_{\epsilon_1})|\exp(z_j\partial_{\epsilon_2})|\chi(\lambda+z_j,\nu_k))_{1\le j\le N, 3\le k\le N}\sum_{i,j=1}^{2} F_{i.j}(\lambda+\epsilon_1,\lambda+\epsilon_2)|_{\epsilon_1=\epsilon_2=0}. \end{aligned} \tag{35}$$

Now let us set $\nu_j = \nu + w_j$ and take the homogeneous limit $\lambda_j \to \lambda, \nu_j \to \nu$ by putting $z_j, w_j, j = 1, \dots, N$ to zero in the order $w_1, w_3, \dots w_N, w_2, z_1 \dots z_N$. We have

$$\begin{aligned} &\Psi_1(M,L)^{(h)} \\ =& \frac{(N-1)!(N-2)!\mathrm{sh}^2\eta\mathrm{sh}^{2N-3}2\nu}{\det_N \Phi\left[\mathrm{sh}^2\nu-\mathrm{sh}^2(\lambda-\eta/2)\right]\left[\mathrm{sh}^2(\nu-\eta/2)-\mathrm{sh}^2\lambda\right]^{L+M-1}\left[\mathrm{sh}^2(\nu+\eta/2)-\mathrm{sh}^2\lambda\right]^{2N-L-M+1}} \\ &\times \det(\partial_{\epsilon_1}^{j-1}|\partial_{\epsilon_2}^{j-1}|\Phi_{j,k-2})_{1\le j\le N, 3\le k\le N}\sum_{i,j=1}^{2} F_{i.j}^{(h)}(\epsilon_1,\epsilon_2)|_{\epsilon_1=\epsilon_2=0}, \end{aligned} \tag{36}$$

where

$$\Phi_{j,k} = \partial_\lambda^{j-1}\partial_\nu^{k-1}\chi(\lambda,\nu), \tag{37}$$

and

$$F_{1,1}^{(h)}(\epsilon_1,\epsilon_2)=\frac{G_1^{(h)}(\epsilon_1)H_1^{(h)}(\epsilon_2)}{\mathrm{sh}^2(\lambda+\epsilon_1+\eta)-\mathrm{sh}^2(\lambda+\epsilon_2)}, \tag{38}$$

$$F_{1,2}^{(h)}(\epsilon_1,\epsilon_2)=\frac{G_1^{(h)}(\epsilon_1)H_2^{(h)}(\epsilon_2)}{\mathrm{sh}^2(\lambda+\epsilon_1+\eta)-\mathrm{sh}^2(\lambda+\epsilon_2)}, \tag{39}$$

$$F_{2,1}^{(h)}(\epsilon_1,\epsilon_2)=\frac{G_2^{(h)}(\epsilon_1)H_1^{(h)}(\epsilon_2)}{\mathrm{sh}^2(\lambda+\epsilon_1-\eta)-\mathrm{sh}^2(\lambda+\epsilon_2)}, \tag{40}$$

$$F_{2,2}^{(h)}(\epsilon_1,\epsilon_2)=\frac{G_2^{(h)}(\epsilon_1)H_2^{(h)}(\epsilon_2)}{\mathrm{sh}^2(\lambda+\epsilon_1-\eta)-\mathrm{sh}^2(\lambda+\epsilon_2)}, \tag{41}$$

where $G_1^{(h)}(\epsilon_1), G_2^{(h)}(\epsilon_1), H_1^{(h)}(\epsilon_2)$ and $H_2^{(h)}(\epsilon_2)$ are

$$G_1^{(h)}(\epsilon_1)=\frac{\mathrm{sh}(\lambda+\epsilon_1+\eta/2-\zeta_+)}{\mathrm{sh}(2\lambda+2\epsilon_1)\mathrm{sh}(2\lambda+\epsilon_1+\eta)}\frac{\left[\mathrm{sh}^2(\lambda+\epsilon_1+\eta)-\mathrm{sh}^2\lambda\right]^L\left[\mathrm{sh}^2(\lambda+\epsilon_1)-\mathrm{sh}^2\lambda\right]^{N-L}}{\left[\mathrm{sh}^2\nu-\mathrm{sh}^2(\lambda+\epsilon_1+\eta/2)\right]^{N-2}}, \tag{42}$$

$$G_2^{(h)}(\epsilon_1)=\frac{\mathrm{sh}(2\lambda+2\epsilon_1+\eta)\mathrm{sh}(\lambda+\epsilon_1-\eta/2+\zeta_+)}{\mathrm{sh}(2\lambda+2\epsilon_1)\mathrm{sh}(2\lambda+2\epsilon_1-\eta)\mathrm{sh}(\epsilon_1-\eta)}\frac{\left[\mathrm{sh}^2(\lambda+\epsilon_1-\eta)-\mathrm{sh}^2\lambda\right]^L\left[\mathrm{sh}^2(\lambda+\epsilon_1)-\mathrm{sh}^2\lambda\right]^{N-L}}{\left[\mathrm{sh}^2\nu-\mathrm{sh}^2(\lambda+\epsilon_1-\eta/2)\right]^{N-2}}, \tag{43}$$

$$\begin{aligned}H_1^{(h)}(\epsilon_2)=&\frac{\mathrm{sh}(2\lambda+\epsilon_2)\mathrm{sh}(\lambda+\epsilon_2+\eta/2-\zeta_+)\mathrm{sh}(\eta/2-\lambda-\epsilon_2-\nu)}{\mathrm{sh}(2\lambda+2\epsilon_2)\mathrm{sh}(\lambda+\epsilon_2+\nu+\eta/2)}\\&\times\frac{\left[\mathrm{sh}^2(\lambda+\epsilon_2+\eta)-\mathrm{sh}^2\lambda\right]^{M-1}\left[\mathrm{sh}^2(\lambda+\epsilon_2)-\mathrm{sh}^2\lambda\right]^{N-M}}{\left[\mathrm{sh}^2\nu-\mathrm{sh}^2(\lambda+\epsilon_2+\eta/2)\right]^{N-2}},\end{aligned} \tag{44}$$

$$\begin{aligned}H_2^{(h)}(\epsilon_2)=&\frac{\mathrm{sh}(2\lambda+2\epsilon_2+\eta)\mathrm{sh}\epsilon_2\mathrm{sh}(-\lambda-\epsilon_2+\eta/2-\zeta_+)\mathrm{sh}(\lambda+\epsilon_2-\nu+\eta/2)}{\mathrm{sh}(2\lambda+2\epsilon_2)\mathrm{sh}(2\lambda+2\epsilon_2-\eta)\mathrm{sh}(\lambda+\epsilon_2-\nu-\eta/2)}\\&\times\frac{\left[\mathrm{sh}^2(\lambda+\epsilon_2-\eta)-\mathrm{sh}^2\lambda\right]^{M-1}\left[\mathrm{sh}^2(\lambda+\epsilon_2)-\mathrm{sh}^2\lambda\right]^{N-M}}{\left[\mathrm{sh}^2\nu-\mathrm{sh}^2(\lambda+\epsilon_2-\eta/2)\right]^{N-2}}.\end{aligned} \tag{45}$$

4 Type II

In this section, we calculate another type of two point function which we call Type II

$$\Psi_2(M,L)=\frac{\psi_2(M,L)}{Z_{2N\times N}(\{\lambda\},\{\nu\})}, \tag{46}$$

where

$$\begin{aligned}\psi_2(M,L)=&w_N^-\mathcal{E}_L(\lambda_N,\{\nu\})\mathcal{B}(\lambda_{N-1},\{\nu\})\cdots\mathcal{B}(\lambda_{M+1},\{\nu\})\\&\times\mathcal{F}(\lambda_M,\{\nu\})\mathcal{B}(\lambda_{M-1},\{\nu\})\cdots\mathcal{B}(\lambda_1,\{\nu\})w_N^+,\end{aligned} \tag{47}$$

$$\mathcal{E}_L(\lambda_\alpha,\{\nu\})=\downarrow_{2\alpha}V_L(\lambda_\alpha,\{\nu\})\uparrow_{2\alpha-1}, \tag{48}$$

$$\begin{aligned}V_L(\lambda_\alpha,\{\nu\})=&T^{t_\alpha}(\lambda_\alpha,\{\nu\})\frac{1}{2}(1+\sigma_L^3)K_+(\lambda_\alpha)(\sigma_\alpha^2L_{\alpha N}(-\lambda_\alpha,\nu_N)\sigma_\alpha^2)\cdots(\sigma_\alpha^2L_{\alpha L}(-\lambda_\alpha,\nu_L)\sigma_\alpha^2)\\&\times\frac{1}{2}(1+\sigma_\alpha^3)(\sigma_\alpha^2L_{\alpha L-1}(-\lambda_\alpha,\nu_{L-1})\sigma_\alpha^2)\cdots(\sigma_\alpha^2L_{\alpha 1}(-\lambda_\alpha,\nu_1)\sigma_\alpha^2),\end{aligned} \tag{49}$$

This two point function, depicted in Figure 3, gives the probability that the spin on the first column is turned down just on the $(2M-1)$-th row, and the spin on the $(2N)$-th row row is turned down just

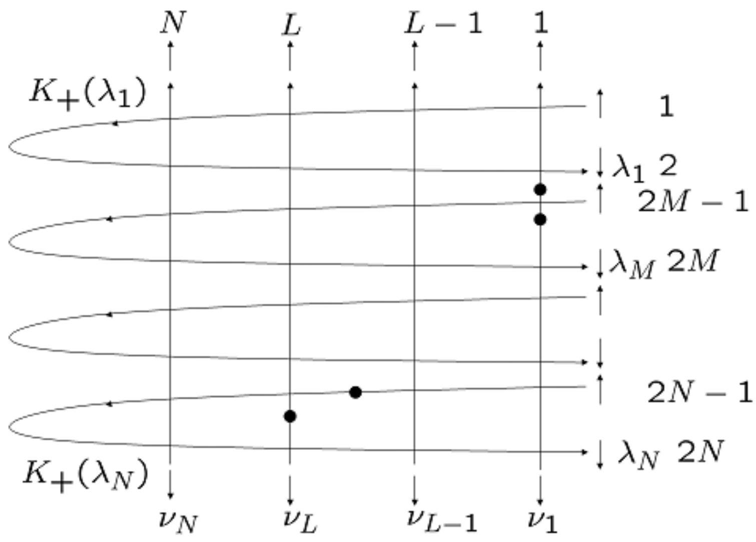

Figure 3: Type II (46).

on the L-th column. Let us calculate this two point function.
First, with the help of the graphical description, we find the numerator $\psi_2(M,L)$ becomes

$$\begin{aligned}\psi_2(M,L) =& \mathrm{sh}(\lambda_N+\eta/2+\zeta_+)b(-\lambda_N-\nu_L-\eta/2)c(\lambda_N-\nu_L-\eta/2)\prod_{j=L+1}^{N} b(\lambda_N-\nu_j-\eta/2)\\ &\times \langle 1,\cdots,\check{L},\cdots,N|\mathcal{B}(\lambda_{N-1},\{\nu\})\cdots\mathcal{B}(\lambda_{M+1},\{\nu\})\\ &\times \mathcal{F}(\lambda_M,\{\nu\})\mathcal{B}(\lambda_{M-1},\{\nu\})\cdots\mathcal{B}(\lambda_1,\{\nu\})w_N^+\\ =& -\mathrm{sh}(\lambda_N+\eta/2+\zeta_+)b(-\lambda_N-\nu_L-\eta/2)c(\lambda_N-\nu_L-\eta/2)\prod_{j=L+1}^{N} b(\lambda_N-\nu_j-\eta/2)\\ &\times c(\lambda_M-\nu_1-\eta/2)\prod_{j=1}^{M-1}\{b(-\lambda_j-\nu_1-\eta/2)b(\lambda_j-\nu_1-\eta/2)\}\\ &\times \langle 1,\cdots,\check{L},\cdots,N-1|\mathcal{B}(\lambda_{N-1},\{\nu\}\backslash\nu_1)\cdots\mathcal{B}(\lambda_{M+1},\{\nu\}\backslash\nu_1)\\ &\times \mathcal{D}(\lambda_M,\{\nu\}\backslash\nu_1)\mathcal{B}(\lambda_{M-1},\{\nu\}\backslash\nu_1)\cdots\mathcal{B}(\lambda_1,\{\nu\}\backslash\nu_1)w_{N-1}^+,\end{aligned} \tag{50}$$

where $\langle 1,\cdots,\check{L},\cdots,n| = \downarrow_1\cdots\uparrow_L\cdots\downarrow_n$. Next, utilizing (20) and

$$\begin{aligned}\prod_{j=1}^{i}\mathcal{B}(\lambda_j,\{\nu\})w_N^+ =& \prod_{j=1}^{i}\frac{\mathrm{sh}(2\lambda_j+\eta)}{\mathrm{sh}(2\lambda_j)}\sum_{\sigma_1=\pm}\cdots\sum_{\sigma_i=\pm}\prod_{j=1}^{i}\{(-\sigma_j)\mathrm{sh}(-\sigma_j\lambda_j+\eta/2-\zeta_+)\}\\ &\times\prod_{j=1}^{i}\prod_{k=1}^{N}\frac{\mathrm{sh}(-\sigma_j\lambda_j-\nu_k-\eta/2)}{\mathrm{sh}(-\sigma_j\lambda_j-\nu_k+\eta/2)}\prod_{1\le j<k\le i}\frac{\mathrm{sh}(\sigma_j\lambda_j+\sigma_k\lambda_k-\eta)}{\mathrm{sh}(\sigma_j\lambda_j+\sigma_k\lambda_k)}\prod_{j=1}^{i}B(\sigma_j\lambda_j,\{\nu\})w_N^+,\end{aligned} \tag{51}$$

(see [27] for the rational case), one finds (50) can be expressed in terms of one-row monodromy matrices as

$$
\begin{aligned}
\psi_2(M,L)=&-\mathrm{sh}\eta\,\mathrm{sh}(\lambda_N+\eta/2+\zeta_+)b(-\lambda_N-\nu_L-\eta/2)c(\lambda_N-\nu_L-\eta/2)c(\lambda_M-\nu_1-\eta/2)\\
&\times\prod_{j=L+1}^{N}b(\lambda_N-\nu_j-\eta/2)\prod_{j=1}^{M-1}\{b(-\lambda_j-\nu_1-\eta/2)b(\lambda_j-\nu_1-\eta/2)\}\prod_{j=1}^{N-1}\frac{\mathrm{sh}(2\lambda_j+\eta)}{\mathrm{sh}(2\lambda_j)}\\
&\times\sum_{\alpha=1}^{M}\sum_{\sigma_1=\pm}\cdots\sum_{\sigma_{N-1}=\pm}\prod_{j=1}^{N-1}\{(-\sigma_j)\mathrm{sh}(-\sigma_j\lambda_j+\eta/2-\zeta_+)\}\prod_{j=1}^{N-1}\prod_{k=2}^{N}\frac{\mathrm{sh}(-\sigma_j\lambda_j-\nu_k-\eta/2)}{\mathrm{sh}(-\sigma_j\lambda_j-\nu_k+\eta/2)}\\
&\times\frac{\mathrm{sh}(-\sigma_\alpha\lambda_\alpha+\lambda_M)}{\mathrm{sh}^2\lambda_M-\mathrm{sh}^2(-\sigma_\alpha\lambda_\alpha+\eta)}\prod_{\substack{j=1\\j\neq\alpha}}^{M}\frac{\mathrm{sh}^2(-\sigma_\alpha\lambda_\alpha+\eta)-\mathrm{sh}^2\lambda_j}{\mathrm{sh}^2\lambda_\alpha-\mathrm{sh}^2\lambda_j}\prod_{\substack{1\le j<k\le N-1\\j,k\neq\alpha}}\frac{\mathrm{sh}(\sigma_j\lambda_j+\sigma_k\lambda_k-\eta)}{\mathrm{sh}(\sigma_j\lambda_j+\sigma_k\lambda_k)}\\
&\times\langle 1\cdots\check{L}\cdots N-1|\prod_{\substack{j=1\\j\neq\alpha}}^{N-1}B(\sigma_j\lambda_j,\{\nu\}\backslash\nu_1)w_{N-1}^{+}.
\end{aligned}\tag{52}
$$

We then change the viewpoint to use the column monodromy matrix

$$
\begin{aligned}
\overline{T}(\nu_j,\{\lambda\})=&L_{Nj}(\lambda_N,\nu_j)\cdots L_{1j}(\lambda_1,\nu_j)\\
=&\begin{pmatrix}\overline{A}(\nu_j,\{\lambda\}) & \overline{B}(\nu_j,\{\lambda\})\\ \overline{C}(\nu_j,\{\lambda\}) & \overline{D}(\nu_j,\{\lambda\})\end{pmatrix},
\end{aligned}\tag{53}
$$

instead of the row transfer matrix (5). Then one finds

$$
\begin{aligned}
&\langle 1\cdots\check{L}\cdots N-1|\prod_{\substack{j=1\\j\neq\alpha}}^{N-1}B(\sigma_j\lambda_j,\{\nu\}\backslash\nu_1)w_{N-1}^{+}\\
=&v_{N-2}^{+}\overline{C}(\nu_N,\sigma(\{\lambda\}\backslash\{\lambda_\alpha,\lambda_N\}))\cdots\overline{C}(\nu_{L+1},\sigma(\{\lambda\}\backslash\{\lambda_\alpha,\lambda_N\}))\\
&\times\overline{A}(\nu_L,\sigma(\{\lambda\}\backslash\{\lambda_\alpha,\lambda_N\}))\overline{C}(\nu_{L-1},\sigma(\{\lambda\}\backslash\{\lambda_\alpha,\lambda_N\}))\cdots\overline{C}(\nu_2,\sigma(\{\lambda\}\backslash\{\lambda_\alpha,\lambda_N\}))v_{N-2}^{-}\\
=&v_{N-2}^{-}\overline{B}(\nu_N,\sigma(\{\lambda\}\backslash\{\lambda_\alpha,\lambda_N\}))\cdots\overline{B}(\nu_{L+1},\sigma(\{\lambda\}\backslash\{\lambda_\alpha,\lambda_N\}))\\
&\times\overline{D}(\nu_L,\sigma(\{\lambda\}\backslash\{\lambda_\alpha,\lambda_N\}))\overline{B}(\nu_{L-1},\sigma(\{\lambda\}\backslash\{\lambda_\alpha,\lambda_N\}))\cdots\overline{B}(\nu_2,\sigma(\{\lambda\}\backslash\{\lambda_\alpha,\lambda_N\}))v_{N-2}^{+}
\end{aligned}\tag{54}
$$

where $v_{N-2}^{+}=\prod_{\alpha=1}^{N-2}\uparrow_\alpha$, $v_{N-2}^{-}=\prod_{\alpha=1}^{N-2}\downarrow_\alpha$ and $\sigma(\{\lambda\}\backslash\{\lambda_\alpha,\lambda_N\})=\{\sigma_1\lambda_1,\dots,\sigma_{\alpha-1}\lambda_{\alpha-1},\sigma_{\alpha+1}\lambda_{\alpha+1},\dots,\sigma_{N-1}\lambda_{N-1}\}$. Utilizing

$$
\begin{aligned}
&\overline{D}(\nu_i,\{\lambda\})\overline{B}(\nu_{i-1},\{\lambda\})\cdots\overline{B}(\nu_1,\{\lambda\})v_N^{+}\\
=&\sum_{k=1}^{i}\frac{\mathrm{sh}\eta}{\mathrm{sh}(\nu_i-\nu_k+\eta)}\prod_{\substack{j=1\\j\neq k}}^{i}\frac{\mathrm{sh}(\nu_j-\nu_k+\eta)}{\mathrm{sh}(\nu_j-\nu_k)}\prod_{j=1}^{N}\frac{\mathrm{sh}(\lambda_j-\nu_k-\eta/2)}{\mathrm{sh}(\lambda_j-\nu_k+\eta/2)}\\
&\times\overline{B}(\nu_i,\{\lambda\})\cdots\check{\overline{B}}(\nu_k,\{\lambda\})\cdots\overline{B}(\nu_1,\{\lambda\})v_N^{+},
\end{aligned}\tag{55}
$$

and the determinant representation of the partition function of the six vertex model on a $N\times N$ lattice with domain wall boundary condition

$$
v_N^{-}\prod_{j=1}^{N}\overline{B}(\nu_j,\{\lambda\})v_N^{+}=\frac{\prod_{j,k=1}^{N}\mathrm{sh}(\lambda_j-\nu_k-\eta/2)\det_N\phi(\{\lambda\},\{\nu\})}{\prod_{1\le j<k\le N}\mathrm{sh}(\nu_j-\nu_k)\prod_{1\le j<k\le n}\mathrm{sh}(\lambda_k-\lambda_j)},\tag{56}
$$

where ϕ is an $N \times N$ matrix whose elements are

$$\phi_{jk} = \phi(\lambda_j, \nu_k), \tag{57}$$

$$\phi(\lambda, \nu) = \frac{\mathrm{sh}\eta}{\mathrm{sh}(\lambda - \nu + \eta/2)\mathrm{sh}(\lambda - \nu - \eta/2)}, \tag{58}$$

one can evaluate (54) as [26]

$$\begin{aligned}
&v_{N-2}^{-}\overline{B}(\nu_N, \sigma(\{\lambda\}\backslash\{\lambda_\alpha, \lambda_N\})) \cdots \overline{B}(\nu_{L+1}, \sigma(\{\lambda\}\backslash\{\lambda_\alpha, \lambda_N\})) \\
&\times \overline{D}(\nu_L, \sigma(\{\lambda\}\backslash\{\lambda_\alpha, \lambda_N\}))\overline{B}(\nu_{L-1}, \sigma(\{\lambda\}\backslash\{\lambda_\alpha, \lambda_N\})) \cdots \overline{B}(\nu_2, \sigma(\{\lambda\}\backslash\{\lambda_\alpha, \lambda_N\}))v_{N-2}^{+} \\
&= \frac{\prod_{\substack{j=1 \\ j\neq\alpha}}^{N-1} \prod_{k=2}^{N} \mathrm{sh}(\sigma_j\lambda_j - \nu_k - \eta/2) \det_{N-1} h(\{\nu\}\backslash\nu_1, \sigma(\{\lambda\}\backslash\{\lambda_\alpha, \lambda_N\}))}{\prod_{2\leq j<k\leq N} \mathrm{sh}(\nu_k - \nu_j) \prod_{\substack{1\leq j<k\leq N-1 \\ j,k\neq\alpha}} \mathrm{sh}(\sigma_j\lambda_j - \sigma_k\lambda_k)}.
\end{aligned} \tag{59}$$

Here, h is an $(N-1) \times (N-1)$ matrix whose elements are given by

$$h_{jk} = \begin{cases} \frac{\prod_{i=2}^{L-1} \mathrm{sh}(\nu_i - \nu_{k+1} + \eta) \prod_{i=L+1}^{N} \mathrm{sh}(\nu_i - \nu_{k+1})}{\prod_{i=1, i\neq\alpha}^{N-1} \mathrm{sh}(\sigma_i\lambda_i - \nu_{k+1} + \eta/2)}, & j = 1, \\ \phi(\sigma_{j-1}\lambda_{j-1}, \nu_{k+1}), & j = 2, \cdots, \alpha, \\ \phi(\sigma_j\lambda_j, \nu_{k+1}), & j = \alpha+1, \cdots, N-1. \end{cases} \tag{60}$$

Combining (8), (52), (54) and (59), we can express $\Psi_2(M, L)$ using determinants as

$$\begin{aligned}
&\Psi_2(M, L) \\
=&\frac{\mathrm{sh}^2\eta\mathrm{sh}(\lambda_N + \eta/2 + \zeta_+)\mathrm{sh}(-\lambda_N - \nu_L - \eta/2)}{\det_N \chi(\{\lambda\}, \{\nu\})\mathrm{sh}(\lambda_M - \nu_1 + \eta/2) \left[\mathrm{sh}^2(\nu_L - \eta/2) - \mathrm{sh}^2\lambda_N\right]} \prod_{j=1}^{N-1} \frac{\mathrm{sh}(2\lambda_j + \eta)}{\mathrm{sh}(2\lambda_j)} \\
&\times \frac{\prod_{k=2}^{N}(\mathrm{sh}^2\nu_k - \mathrm{sh}^2\nu_1) \prod_{2\leq j<k\leq N} \mathrm{sh}(\nu_k + \nu_j) \prod_{j=1}^{N-1}(\mathrm{sh}^2\lambda_j - \mathrm{sh}^2\lambda_N)}{\prod_{j=1}^{M-1} \left[\mathrm{sh}^2(\nu_1 - \eta/2) - \mathrm{sh}^2\lambda_j\right] \prod_{j=M}^{N} \left[\mathrm{sh}^2(\nu_1 + \eta/2) - \mathrm{sh}^2\lambda_j\right]} \\
&\times \frac{1}{\prod_{j=2}^{L} \left[\mathrm{sh}^2(\nu_j + \eta/2) - \mathrm{sh}^2\lambda_N\right] \prod_{j=L+1}^{N} \left[\mathrm{sh}^2\nu_j - \mathrm{sh}^2(\lambda_N + \eta/2)\right]} \\
&\times \sum_{\alpha=1}^{M} \sum_{\sigma_1=\pm} \cdots \sum_{\sigma_{N-1}=\pm} \prod_{j=1}^{N-1} \{(-\sigma_j)\mathrm{sh}(-\sigma_j\lambda_j + \eta/2 - \zeta_+)\} \frac{\prod_{1\leq j<k\leq N-1} \mathrm{sh}(\sigma_j\lambda_j + \sigma_k\lambda_k - \eta)}{\prod_{j=1}^{N-1} \prod_{k=2}^{N} \mathrm{sh}(-\sigma_j\lambda_j - \nu_k + \eta/2)} \\
&\times \frac{\prod_{j=M+1}^{N-1}(\mathrm{sh}^2\lambda_\alpha - \mathrm{sh}^2\lambda_j) \prod_{j=1}^{M-1} \mathrm{sh}(\sigma_\alpha\lambda_\alpha - \sigma_j\lambda_j - \eta)}{\prod_{j=2}^{N} \mathrm{sh}(\sigma_\alpha\lambda_\alpha - \nu_j - \eta/2) \prod_{j=M}^{N-1} \mathrm{sh}(\sigma_\alpha\lambda_\alpha + \sigma_j\lambda_j - \eta)} \\
&\times (-1)^\alpha \mathrm{sh}(-\sigma_\alpha\lambda_\alpha + \lambda_M) \det_{N-1} h(\{\nu\}\backslash\nu_1, \sigma(\{\lambda\}\backslash\{\lambda_\alpha, \lambda_N\})).
\end{aligned} \tag{61}$$

5 Conclusion

In this paper, we considered two point functions for the six vertex model on a $2N \times N$ lattice with domain wall boundary condition and left reflecting boundary. We calculated two types: (Type I) the probability that the spin on the first column is turned down just on the $(2M-1)$-th row, and the spin on the second column is turned down just on the $(2L)$-th row, (Type II) the probability that the spin on the first column is turned down just on the $(2M-1)$-th row, and the spin on the $(2N)$-th row is turned down just on the L-th column. For Type I, we could express it utilizing a single determinant whose entries contain differential operators which act on some functions. For Type II, reducing the double row monodromy matrices to one row monodromy matrices and then changing the viewpoint to use column monodromy matrices, we expressed it as combinations of determinants.

It is interesting to extend the analysis to multipoint functions such as the emptiness formation probability. Another interesting direction is to study one point functions of higher rank/spin generalized vertex models with domain wall boundary condition. The thorough investigation of the off shell structure [28, 29] should be crucial for the analysis.

References

[1] J.C. Slater, J. Chem. Phys. **9** (1941) 16.

[2] E.H. Lieb, Phys. Rev. **162** (1967) 162.

[3] B. Sutherland, Phys. Rev. Lett. **19** (1967) 103.

[4] R.J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic press, San Diego, 1982).

[5] V.E. Korepin, Commun. Math. Phys. **86** (1982) 391.

[6] N.A. Slavnov, Theor. Math. Phys. **79** (1989) 502.

[7] A.G. Izergin, Sov. Phys. Dokl. **32** (1987) 878.

[8] A.G. Izergin, D.A. Coker and V.E. Korepin, J. Phys. A **25** (1992) 4315.

[9] V.E. Korepin, A.G. Izergin, F.H.L Essler and D.B. Uglov, Phys. Lett. A **190** (1994) 182.

[10] V.E. Korepin, N.M. Bogoliubov and A.G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge University Press, Cambridge, 1993).

[11] N. Kitanine, J.M. Maillet and V. Terras, Nucl. Phys. B **567** (2000) 554.

[12] N. Kitanine, J.M. Maillet, N.A. Slavnov and V. Terras, Nucl. Phys. B **641** (2002) 487.

[13] D. Zeilberger, Elec. J. Comb. **3** (2) (1996) R13.

[14] G. Kuperberg, Int. Math. Res. Not. **1996** (1996) 139.

[15] D.M. Bressoud, *Proofs and Confirmations: The Story of the Alternating Sign Matrix Conjecture* (Cambridge University Press, Cambridge, 1999).

[16] A. Lascoux, SIGMA **3** (2007) 029.

[17] O. Foda, M. Wheeler and M. Zuparic, J. Stat. Mech. (2009) P03017.

[18] N.M. Boboliubov, A.V. Kitaev and M.B. Zvonarev, Phys. Rev. E **65** (2002) 026126.

[19] N.M. Bogoliubov, A.G. Pronko and M.B. Zvonarev, J. Phys. A **35** (2002) 5525.

[20] O. Foda and I. Preston, J. Stat. Mech. (2004) P11001.

[21] F. Colomo and A.G. Pronko, J. Stat. Mech. (2005) P05010.

[22] F. Colomo and A.G. Pronko, Nucl. Phys. B **798** (2008) 340.

[23] E.K. Sklyanin, J. Phys. A **21** (1998) 2375.

[24] O. Tsuchiya, J. Math. Phys. **39** (1998) 5946.

[25] Y-S. Wang, J. Phys. A **36** (2003) 4007.

[26] K. Motegi, arXiv:1005.5037.

[27] Y-S. Wang, Nucl. Phys. B **622** (2002) 633.

[28] C.S. Melo and M.J. Martins, Nucl. Phys. B **806** (2009) 567.

[29] M.J. Martins and C.S. Melo, Nucl. Phys. B **820** (2009) 620.